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On Minimal and Quasi-Minimal Supported Bivariate Splines

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1. INTRODUCTION

Let Δ be a grid partition of the plane $\mathbb{R}^2$ with grid lines $x=i$, $y=i$, $x+y=i$, and $x-y=i$, where $i=..., -1, 0, 1, ...$. This partition is sometimes called a type-2 or uniform criss-cross triangulation, and it forms a 4-direction mesh. Let d and k be nonnegative integers. We denote by $S_d^k(\Delta)$ the space of all functions in $C^k(\mathbb{R}^2)$ whose restriction to each triangular cell of the partition Δ is a restriction of an element of π_d, the space of all polynomials in x and y of total degree at most d. $S_d^k(\Delta)$ is called a space of bivariate splines. Clearly, for $S_d^k(\Delta)$ to be nonempty, we must have $d>k$. We will only study the case where d is the smallest integer $m(k)$ such that $S_d^k(\Delta)$ contains at least one locally supported (ls) function f; and by this, we mean that f is not identically zero, but vanishes outside a compact set. It is well known that $m(3r)=4r+1$, $m(3+1)=4r+2$, and $m(3r+2)=4r+4$, $r=0, 1, ...$ (cf. [5] and [9]). The support of an ls function f in $S_d^k(\Delta)$ is the closure of the set on which f does not vanish and is denoted by supp(f). A set S is called a minimal support of $S_d^k(\Delta)$ if there is some f in $S_d^k(\Delta)$ with supp(f) $=S$, but there does not exist a nontrivial g in $S_d^k(\Delta)$ with supp(g) properly contained in S. For both theoretical and application purposes, it is important to determine all functions in $S_d^k(\Delta)$ with minimal supports, and these functions will be called minimal supported (ms) bivariate splines.

Let $\Omega=[0, n_1]\otimes[0, n_2]$, where n_1 and n_2 are are positive integers, and denote the space of all restrictions of functions in $S_d^k(\Delta)$ on Ω by $S_d^k(\Delta, \Omega)$. It is also well known (cf. [9]) that

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$$\dim S^{3r}_{4r+1}(\Delta, \Omega) = 2n_1 n_2 + \left[3\binom{r+2}{2} - 2\right](n_1 + n_2)$$
$$+ \binom{4r+3}{2} - 4\binom{r+2}{2} + 2,$$
$$\dim S^{3r+1}_{4r+2}(\Delta, \Omega) = n_1 n_2 + \left[3\binom{r+2}{2} - 1\right](n_1 + n_2)$$
$$+ \binom{4r+4}{2} - 4\binom{r+2}{2} + 1,$$

and

$$\dim S^{3r+2}_{4r+4}(\Delta, \Omega) = 3n_1 n_2 + \left[3\binom{r+3}{2} - 3\right](n_1 + n_2)$$
$$+ \binom{4r+6}{2} - 4\binom{r+3}{2} + 3.$$

Hence, the spaces $S^{3r}_{4r+1}(\Delta)$, $S^{3r+1}_{4r+2}(\Delta)$, and $S^{3r+2}_{4r+4}(\Delta)$ should have two, one, and three "linearly independent" ls bivariate splines, respectively. In [5], de Boor and Höllig constructed two ms bivariate splines in $S^{3r}_{4r+1}(\Delta)$ and one ms one in $S^{3r+1}_{4r+2}(\Delta)$. It is therefore natural to expect that $S^{3r+2}_{4r+4}(\Delta)$ has three ms functions. In this paper, we will construct two ms bivariate splines in $S^{3r+2}_{4r+4}(\Delta)$ and prove, surprisingly, that every ms function in this space with convex support is some constant multiple of a translate of one of them. Hence, the notion of quasi-minimal supported (qms) bivariate splines is introduced as follows:

A function f in $S^k_d(\Delta)$ will be called a qms bivariate spline if

(i) f cannot be written as a (finite) linear combination of ms bivariate splines in $S^k_d(\Delta)$, and

(ii) for any h in $S^k_d(\Delta)$ with $\mathrm{supp}(h)$ properly contained in $\mathrm{supp}(f)$, h is some (finite) linear combination of ms bivariate splines in $S^k_d(\Delta)$.

We will construct a qms bivariate spline in $S^{3r+2}_{4r+4}(\Delta)$, and prove that any qms function in this space with convex support is some constant multiple of a translate of this function modulo a ms function. Hence, together with the two ms bivariate splines, these three functions are the "unique" ls ones with "smallest" possible convex supports, and it can be shown that they generate all ls bivariate splines in this space. We remark that when the definition of minimal support was given in [5], the word "uniqueness" was used but no further discussion was included. In fact, the problem of uniqueness is still open if the convexity assumption is not imposed. We will use the ntation $lS^k_d(\Delta)$ to denote the subspace of all ls functions in $S^k_d(\Delta)$, and $lS_d^k(\Delta, \Omega)$ the subspace of $S_d^k(\Delta, \Omega)$ that consists of all restrictions of $lS_d^k(\Delta)$ on Ω. We will show that the three ls bivariate splines (two ms and one qms) in $S_{4r+4}^{3r+2}(\Delta)$ form a spanning set of $lS_{4r+4}^{3r+2}(\Delta, \Omega)$, but somewhat surprisingly, $lS_{4r+4}^{3r+2}(\Delta, \Omega)$ is all of $S_{4r+4}^{3r+2}(\Delta, \Omega)$ if and only if $r=0$ or 1. Hence, if $r \geqslant 2$, some functions in $S_{4r+4}^{3r+2}(\Delta, \Omega)$ cannot be locally produced. These results were announced in [6]. In addition, some algebraic and approximation properties of the ms and qms bivariate splines in $S_{4r+4}^{3r+2}(\Delta)$ will also be included in this paper.

To study minimal and quasi-minimal supports, we need the following notation. If f is an ls function in $S_d^k(\Delta)$ whose support is a convex polygon, it is clear that none of its vertices lies on the intersection of two grid lines, and we will denote its support by

$$\operatorname{supp}(f) = \{d_1, d_2, ..., d_8\},$$

where $d_1, d_2, ..., d_8$ are nonnegative integers, indicating the number of units (i.e., horizontal or vertical edges, or diagonals) of the partition Δ in the "directions" $e_1 - e_2, e_1, ..., -e_2$, respectively, as shown in Fig. 1 below. Here, and throughout, we use the usual notation $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

For convenience, we will also use the notation

$$S_r = S_{4r+4}^{3r+2}(\Delta) \qquad S_r(Q) = S_{4r+4}^{3r+2}(\Delta, \Omega)$$
$$lS_r = lS_{4r+4}^{3r+2}(\Delta) \qquad lS_r(\Omega) = lS_{4r+4}^{3r+2}(\Delta, \Omega).$$

It will be seen that the special case $r=0$ is particularly important since the general case is based on this example. Hence, we devote Section 3 to this special case. In Section 2 we discuss some preliminary results on cone splines (or truncated powers) which will be used as tools to prove that certain supports are minimal or quasi-minimal. The general case of S_r is studied in Section 4.

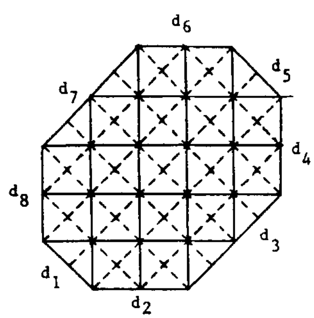


FIGURE 1

2. Preliminary Results

In this section we give explicit formulation of two bases of $S_r(\Omega)$. These bases have been introduced in [9] where certain linear equations must be solved. In the multivariate setting, the functions we discuss here are called truncated powers in [11] and cone splines in [4]. We first consider the case $r=0$. Let

$$A_1(x,y)=\begin{cases} x^3(2y-x) & \text{for } x\geqslant 0 \text{ and } y\geqslant x\\ (2x-y)\,y^3 & \text{for } y\geqslant 0 \text{ and } y<x\\ 0 & \text{otherwise}\end{cases}$$

$$A_2(x,y)=\begin{cases} (x+y)^4 & \text{for } x+y>0 \text{ and } x<0\\ 4y^4+(x-y)^3(5x+3y) & \text{for } x>0 \text{ and } y>x\\ 4y^4 & \text{for } y>0 \text{ and } y<x\\ 0 & \text{otherwise}\end{cases}$$

$$A_3(x,y)=\begin{cases} (y-3x)(x+y)^3 & \text{for } x+y\geqslant 0 \text{ and } x\leqslant 0\\ (y-x)^3(y+3x) & \text{for } x>0 \text{ and } y\geqslant x\\ 0 & \text{otherwise}\end{cases}$$

$$B_1(x,y)=\begin{cases} y(2x+y) & \text{for } y\geqslant 0 \text{ and } x+y\leqslant 0\\ x^3(x+2y) & \text{for } x+y>0 \text{ and } x\leqslant 0\\ 0 & \text{otherwise}\end{cases}$$

$$B_2(x,y)=\begin{cases} 4y^4 & \text{for } y\geqslant 0 \text{ and } x+y\leqslant 0\\ (x-y)^4+x^3(4x+16y) & \text{for } x+y>0 \text{ and } x\leqslant 0\\ (x-y)^4 & \text{for } x>0 \text{ and } y\geqslant x\\ 0 & \text{otherwise}\end{cases}$$

and

$$B_3(x,y)=-A_3(x,y).$$

Also, set $A_k^{ij}(x,y)=A_k(x-i,y-j)$ and $B_k^{ij}(x,y)=B_k(x-i,y-j)$, and consider the collections

$$E^0=E_1\cup E_2\cup E_3 \qquad \text{and} \qquad F^0=F_1\cup F_2\cup F_3,$$

where

$$E_1=\{A_1^{ij}\colon i=-2,\ldots,n_1-1,\, j=-2,\ldots,n_2-1\}$$
$$E_2=\{A_2^{ij}\colon i=-1,\ldots,n_1,\, j=-2,\ldots,n_2-1,$$
$$\text{with } (i,j)\neq(-1,-1),(-1,-2),(0,-2)\}$$

$$E_3 = \{A_3^{ij}: i = 0, \ldots, n_1+1, j = -2, \ldots, n_2-1,$$
$$\text{with } (i,j) \neq (0,-1), (0,-2), (n_1+1, n_2-1)\}$$
$$F_1 = \{B_1^{ij}: i = -1, \ldots, n_1+2, j = -2, \ldots, n_2-1\}$$
$$F_2 = \{B_2^{ij}: i = 0, \ldots, n_1+1, j = -2, \ldots, n_2-1,$$
$$\text{with } (i,j) \neq (n_1+1, -1), (n_1+1, -2), (n_1, -2)\}$$

and

$$F_3 = \{B_3^{ij}: i = -1, \ldots, n_1, j = -2, \ldots, n_2-1,$$
$$\text{with } (i,j) \neq (n_1, -1), (n_1, -2), (-1, n_2-1)\}.$$

We have the following.

LEMMA 1. *Each E^0 and F^0 is a basis of $S_0(\Omega)$.*

Proof. We only consider E^0 since the proof for F^0 is identical. Let

$$f(x,y) = \sum_{k=1}^{3} \sum_{ij} a_k^{ij} A_k^{ij}(x,y),$$

where all elements of E are used. If f vanishes on the triangle with vertices $(0,0)$, $(1,0)$, and $(\frac{1}{2}, \frac{1}{2})$, then we have $M_1\rho_1 = 0$ and $M_2\rho_2 = 0$, where

$$M_1 = \begin{bmatrix} 1 & 1 & -1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 & 2 & 1 & 2 \\ 1 & 1 & 1 & 4 & 2 & 1 & 4 \\ 0 & 2 & 0 & 0 & 2 & 1 & 6 \\ 1 & 1 & 4 & 8 & 2 & 1 & 6 \\ 1 & 4 & 1 & 8 & 2 & 1 & 8 \\ 1 & 3 & 3 & 16 & 1 & 1 & 5 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & 0 & 1 & -9 & -9 \\ 0 & 0 & 0 & 1 & 1 & 0 & 2 & -4 & -4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ -1 & -1 & -1 & 0 & 0 & 4 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix},$$

$\rho_1 = [a_1^{-1,-1} a_1^{-2,-1} a_1^{-1,-2} a_1^{-2,-2} a_2^{0,-1} a_2^{1,-2} a_2^{1,-2} + a_3^{1,-2}]^{\mathrm{T}}$, and $\rho_2 = [a_1^{0,0} a_1^{-1,0} a_1^{-2,0} a_1^{0,-1} a_1^{0,-2} a_2^{0,0} + a_2^{-1,0} a_2^{1,-1} + a_2^{2,-2} a_3^{1,-1} a^{2}{}_3^{-2}]^{\mathrm{T}}$. Using the assumption that f vanishes on the triangle with vertices $(0,0)$, $(0,1)$, and $(\frac{1}{2}, \frac{1}{2})$, we have $M_3\rho_3 = 0$, where

$$M_3 = \begin{bmatrix} -1 & 0 & 0 & -1 & -1 & 0 & 1 & -9 & -9 \\ 1 & 0 & 0 & 1 & 1 & 0 & 2 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 4 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 2 & 4 \end{bmatrix}$$

and $\rho_3 = [a_1^{0,0} a_1^{-1,0} a_1^{-2,0} a_1^{0,-1} a_1^{0,-2} a_2^{0,0} \;+\; a_2^{-1,0} a_2^{1,-1} \;+\; a_2^{2,-2} a_3^{1,-1} a_3^{2,-2}]^{\mathrm{T}}$. Hence, $\rho_1 = \rho_2 = \rho_3 = 0$, and in particular,

$$a_2^{0,0} + a_2^{-1,0} = 0, \qquad a_2^{1,-1} + a_2^{2,-2} = 0.$$

By considering $f(x, y) = 0$ on the triangle with vertices $(0, 1)$, $(\frac{1}{2}, \frac{1}{2})$, and $(1, 1)$, we also have $a_2^{0,0} = 0$ and $a_2^{1,-1} = 0$, so that $a_2^{-1,0} = 0$ and $a_2^{2,-2} = 0$. This takes care of the first unit square. By repeating the same process on each of the squares on the first row, then the second row, and so forth, we may conclude that E^0 is a linearly independent set on Ω, and since its cardinality agrees with the dimension of $S_0(\Omega)$, it forms a basis of this space. This completes the proof of the lemma.

To consider the general case $r > 0$, we introduce the integral operators

$$(J_0 f)(x, y) = \int_{-\infty}^{x} f(s, y)\, ds$$

$$(J_1 f)(x, y) = \int_{x}^{\infty} f(s, y)\, ds$$

$$(J_2 f)(x, y) = \int_{-\infty}^{y} f(x, t)\, dt$$

$$(J_3 f)(x, y) = \int_{-\infty}^{x} f(u, u + y - x)\, du$$

$$(J_4 f)(x, y) = \int_{x}^{\infty} f(v, x + y - v)\, dv$$

and set

$$J = J_4 J_3 J_2 J_0, \qquad \bar{J} = J_4 J_3 J_2 J_1.$$

In addition, let

$$D = D_1 D_2^3 - D_1^3 D_2, \qquad \bar{D} = -D,$$

where, as usual,

$$D_1^l f = \frac{\partial^l}{\partial x^l} f, \qquad D_2^l f = \frac{\partial^l}{\partial y^l} f.$$

Hence, it follows that

$$JD = \bar{J}\bar{D} = I,$$

where I is the identity operator on S_r, $r \geqslant 1$. It is clear that if $s \in S_{r-1}$, then Js and $\bar{J}s$ are in S_r. We also have the following.

LEMMA 2. *For any $s \in S_r$ where $r \geqslant 1$, $Ds = -\bar{D}s$ is in S_{r-1}.*

Proof. Let Ω_1 and Ω_2 be two adjacent cells of the partition Δ with common grid line defined by the linear equation $l(x, y) = 0$. For any $s \in S_r$, let $p_i = s\,|_{\Omega_i}$, $i = 1, 2$. Then since $s \in C^{3r+2}$, we have

$$p_1 - p_2 = l^{3r+3} q,$$

where q is a polynomial of total degree $r + 1$ (cf. [8]). Now, it is clear that

$$\begin{aligned} D(p_1 - p_2) = (3r+3) \cdots (3r)[D_1 l\, D_2^3 l - D_1^3 l\, D_2 l]\, l^{3r-1} q \\ + l^{3(r-1)+3} \bar{q} = l^{3(r-1)+3} \bar{q} \end{aligned}$$

for some polynomial $\bar{q}$ of total degree r. That is, Ds is in $C^{3(r-1)+2}$ in the closure of $\Omega_1 \cup \Omega_2$, and is a polynomial of total degree $3(r-1) + 3 + r = 4(r-1) + 4$. This proves that Ds is in S_{r-1}.

Next, consider the collections

$$E^r = J^r s \colon s \in E^0\}$$

and

$$F^r = \{\bar{J}^r s \colon s \in F^0\}.$$

Clearly, E^r and F^r are subcollections of S_r. In fact, by using Lemma 2 and Lema 1 consecutively, it can be seen that any function in $lS_r(\Omega)$ is a linear combination of functions in E^r, and is also a linear combination of functions in F^r.

We also need the functions

$$A_{i,r} = J^r A_i \qquad \text{and} \qquad B_{i,r} = \bar{J}^r B_i,$$

where $i = 1, 2, 3$. By simple computation, it can be shown that

$$A_{i,r}\,|_{x \leqslant 0 \text{ and } x+y \geqslant 0} = (x+y)^{3r+6-i} p_i$$

$$A_{i,r}\,|_{y \geqslant 0 \text{ and } y-x \leqslant 0} = y^{3r+2+i} \bar{p}_i$$

$$B_{i,r}\,|_{y \geqslant 0 \text{ and } x+y \leqslant 0} = y^{3r+2+i} q_i$$

and

$$B_{i,r}\,|_{x\geqslant 0 \text{ and } y-x\geqslant 0}=(x-y)^{3r+6-i}\bar{q}_i$$

for some polynomials p_i, $\bar{p}_i$, q_i, and $\bar{q}_i$ of total degrees $r-2+i$, $r+2-i$, $r+2-i$, and $r-2+i$, respectively, where $i=1, 2, 3$. Set

$$p_i=p_{i,1}+p_{i,2}$$

$$\bar{p}_i=\bar{p}_{i,1}+\bar{p}_{i,2}$$

$$q_i=q_{i,1}+q_{i,2}$$

and

$$\bar{q}_i=\bar{q}_{i,1}+\bar{q}_{i,2},$$

where $p_{i,1}$, $\bar{p}_{i,1}$, $q_{i,1}$, $\bar{q}_{i,1}$ are homogeneous polynomials of degrees $r-2+i$, $r+2-i$, $r+2-i$, $r-2+i$, respectively, and $p_{i,2}$, $\bar{p}_{i,2}$, $q_{i,2}$, $\bar{q}_{i,2}$ are polynomials of degree $r-3+i$, $r+1-i$, $r+1-i$, $r-3+i$, respectively. We need the following.

LEMMA 3. *For each $i=1, 2, 3$, $p_{i,1}$ is not divisible by $x+y$, $\bar{p}_{i,1}$ and $q_{i,1}$ are not divisible by y, and $\bar{q}_{i,1}$ is not divisible by $x-y$.*

Proof. It is sufficient to prove this result for $\bar{p}_{1,1}$, since the others can be verified in the same manner. Suppose, on the contrary, that

$$\bar{p}_{1,1}(x, y)=yu(x, y)$$

for some polynomial u of degree r. Then

$$\begin{aligned} D^r[y^{3r+3}\bar{p}_1] &= D^r[y^{3r+4}u+y^{3r+3}\bar{p}_{1,2}] \\ &= c_1y^4+c_2y^3 \end{aligned}$$

for some constants c_1 and c_2. On the other hand, we also have

$$\begin{aligned} D^rA_{1,r}\,|_{y\geqslant 0 \text{ and } y-x\leqslant 0} &= A_1\,|_{y\geqslant 0 \text{ and } y-x\leqslant 0} \\ &= y^3(2x-y). \end{aligned}$$

Hence, $2x-y\equiv c_1y+c_2$ which is a contradiction.

3. RESULTS ON S_0

In order to study the general case S_r, it is necessary to establish several results for $r=0$. We first discuss the ls functions s in S_0 with

$\mathrm{supp}(s) = \{d_1, ..., d_8\}$ such that $d_1, ..., d_8$ are the smallest possible. Such bivariate spline functions will be said to have *smallest* 8-*tuple supports.*

LEMMA 4. *The supports of the nontrivial ls functions in* S_0 *with smallest* 8-*tuple supports are*

$$\{1, 1, 1, 1, 1, 1, 1, 1\},$$

or

$$\{0, 2, 0, 2, 0, 2, 0, 2\},$$
$$\{2, 0, 2, 0, 2, 0, 2, 0\}.$$

Proof. Let $s \in S_0$ with $\mathrm{supp}(s) = \{d_1, ..., d_8\}$. We first determine the smallest d_1, d_2, d_3. Without loss of generality, assume that the lower left vertex of $\mathrm{supp}(s)$ is at the origin. Let

$$\hat{\Omega} = [-d_1 - 1, d_2 + 1]] \otimes [-1, d_1 + 1].$$

By Lemma 1, with a simple translation along the x-axis, we may write

$$\begin{aligned} s(x, y) = \sum_{k=1}^{3} \{&[a_{k,d_2} A_k^{d_2,0}(x, y) + a_{k,d_2-1} A_k^{d_2-1,0}(x, y) \\ &+ \cdots + a_{k,0} A_k^{0,0}(x, y)] + [a_{k,-1} A_k^{-1,1}(x, y) + a_{k,-2} A_k^{-2,2}(x, y) \\ &+ \cdots + a_{k,-d_1} A_k^{-d_1 d_1}(x, y)]\} \end{aligned}$$

for all (x, y) in

$$\Lambda = \hat{\Omega} \cap [\{(x, y): x \leqslant 0, 0 \leqslant x + y \leqslant 1\} \cup \{(x, y): 0 \leqslant x \leqslant y + d_2, 0 \leqslant y \leqslant 1\}].$$

Since $s(x, y) = 0$ for all (x, y) in the triangles λ_1 and λ_2 where the vertices of λ_1 are $(-d_1, d_1)$, $(-d_1, d_1 + 1)$, $(-d_1 - \frac{1}{2}, d_1 + \frac{1}{2})$ and those of λ_2 are $(d_2, 0)$, $(d_2 + 1, 0)$, $(d_2 + \frac{1}{2}, \frac{1}{2})$, and $\lambda_1, \lambda_2 \subset \Lambda$, we have the linear system

$$\begin{aligned} a_{1,1} + 2a_{1,2} + \cdots + d_2 a_{1,d_2} &= 0 \\ a_{1,0} + a_{1,1} + \cdots + a_{1,d_2} &= 0 \\ a_{2,0} + a_{2,-1} + \cdots + a_{2,-d_1} &= 0 \\ a_{2,0} + a_{2,1} + \cdots + a_{2,d_2} &= 0 \\ a_{3,-1} + 2a_{3,-2} + \cdots + d_1 a_{3,-d_1} &= 0 \\ a_{3,0} + a_{3,-1} + \cdots + a_{3,-d_1} &= 0. \end{aligned} \tag{1}$$

The smallest nonnegative integers d_1 and d_2 for which the linear system has a nontrivial solution are given by the paris

$$\begin{cases} d_1 = 1 \\ d_2 = 1 \end{cases}, \qquad \begin{cases} d_1 = 0 \\ d_2 = 2 \end{cases}, \qquad \begin{cases} d_1 = 2 \\ d_2 = 0 \end{cases}.$$

Similarly, by using the other basis in Lemma 1, the smallest d_2 and d_3 for a nontrivial solution are the pairs

$$\begin{cases} d_2 = 1 \\ d_3 = 1 \end{cases}, \qquad \begin{cases} d_2 = 2 \\ d_3 = 0 \end{cases}, \qquad \begin{cases} d_2 = 0 \\ d_3 = 2 \end{cases}.$$

Hence, the "smallest" triples (d_1, d_2, d_3) are

$$(1, 1, 1), \qquad (0, 2, 0), \qquad (2, 0, 2).$$

By the symmetry of the grid Δ, we can conclude that the "smallest" 8-tuples are the required ones. This completes the proof of the lemma.

In Figs. 2a, 2b, and 2c, we give the Bézier representation of three functions f_1^0, f_2^0, and f_3^0 in S_0 with supports $\{1, 1, 1, 1, 1, 1, 1, 1\}$, $\{0, 2, 0, 2, 0, 2, 0, 2\}$, and $\{2, 0, 2, 0, 2, 0, 2, 0\}$, respectively. f_1^0 and f_2^0 were constructed by Sablonniere in [16], and f_3^0 was constructed in [7].

It is stragithforward to verify that $\operatorname{supp}(f_1^0)$ and $\operatorname{supp}(f_2^0)$ are minimal, but is not clear if f_1^0 and f_2^0 are the only two ms functions in S_0. At this stage, we can only show that f_1^0 and f_2^0 are the only two ms functions with convex supports in S_0, in the sense that any such function (denoted by "cms function" in S_0) is a constant multiple of some translate of f_1^0 or f_2^0. We conjecture, however, that S_r has no nonconvex supported ms splines. Similarly, we will also show that f_3^0 is "unique" among functions with convex quasi-minimal supports (cqms), and again conjecture that S_r has no qms splines with nonconvex supports.

Throughout, we will use the notation

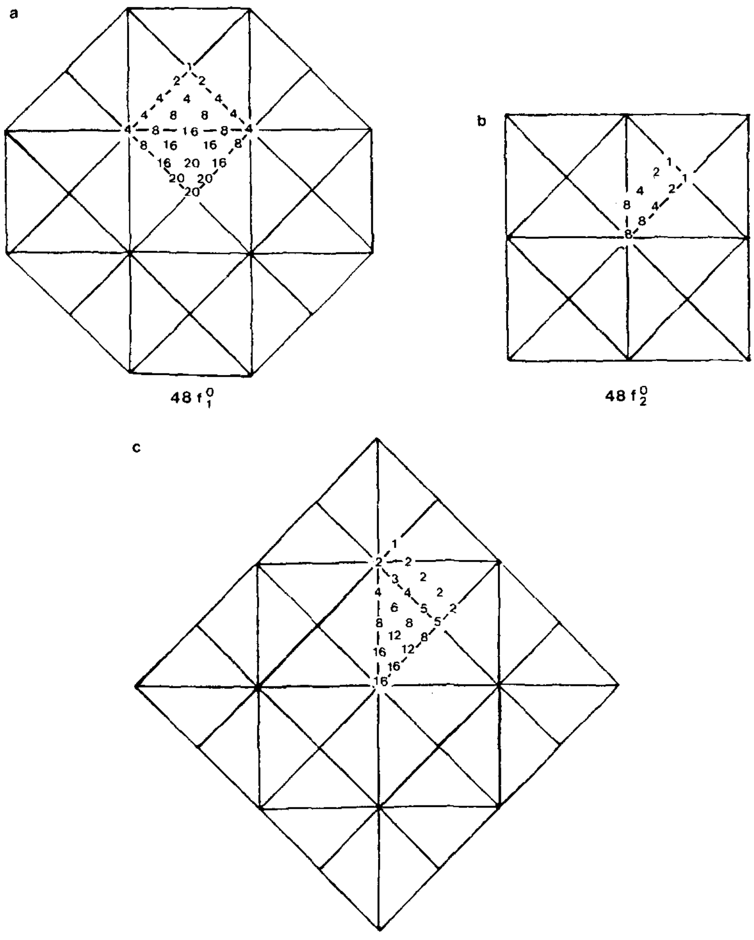
$$f_{i,j}^r(\cdot) = f(\cdot - j)$$

and

$$T_i^r = \{c_j f_{i,j}^r : j \in Z^2,\ c_j \text{ constant}\}, \tag{2}$$

$i = 1, 2, 3$. We have the following result.

Proposition 1. *f_1^0 and f_2^0 are ms functions in S_0 and f_3^0 is a qms function in S_0. Furthermore, f_1^0, f_2^0, and f_3^0 are unique among functions with convex supports in the sense that if f is a cms function in S_0 then $f \in T_1^0 \cup T_2^0$, and if f is cqms in S_0 then $f - g \in T_3^0$ for some $g \in T_2^0$ with* $\operatorname{supp}(g) \subset \operatorname{supp}(f)$.



a: $48\,f_1^0$ b: $48\,f_2^0$ c: $48\,f_3^0$

FIGURE 2

Proof. We first prove that f_1^0 and f_2^0 are ms bivariate splines. Although there is a more direct proof, we introduce the following procedure which can be adopted for the general case S_r. Suppose that $f \in S_0$ with

$$\operatorname{supp}(f) \subset \operatorname{supp}(f_1^0).$$

Let $\operatorname{supp}(f) \subset \{d_1, ..., d_8\} \subset \{1, 1, 1, 1, 1, 1, 1, 1\}$. Since $d_1 + d_2 \geqslant 2$ and $d_8 + d_1 \geqslant 2$, in order that $f \not\equiv 0$, we must have $d_1 = d_2 = d_8 = 1$. Similarly, $d_1 = \cdots = d_8 = 1$. From the linear system (1), we have

$$a_{1,1} = a_{1,0}, \qquad a_{2,0} = -a_{2,1}, \qquad a_{2,-1} = a_{2,1}, \qquad a_{3,-1} = a_{3,0} = 0$$

so that, for $(x, y) \in \Lambda$,

$$f(x, y) = a_{2,1}(A_2^{-1,1} - A_2^{0,0} + A_2^{1,0})(x,y),$$

and this yields $f - cf_1^0 = 0$ on Λ for some constant $c \neq 0$. This shows that $\operatorname{supp}(f - cf_1^0) \subsetneq \operatorname{supp}(f_1^0)$ such that the smallest 8-tuple support of $f - cf_1^0$ is properly contained in $\{1, 1, 1, 1, 1, 1, 1, 1\}$, contradicting Lemma 4, unless $f - cf_1^0 \equiv 0$. This proves not only that f_1^0 is ms, but it is also the unique one with support contained in $\operatorname{supp}(f_1^0)$.

Suppose that $\operatorname{supp}(f) \subset \operatorname{supp}(f_2^0)$, and let $\operatorname{supp}(f) \subset \{d_1, ..., d_8\} \subset \{0, 2, 0, 2, 0, 2, 0, 2\}$. Then by the same argument as above, $\{d_1, ..., d_8\} = \{0\ 2, 0, 2, 0, 2, 0, 2\}$, and for $(x, y) \in \Lambda$,

$$\begin{aligned} f(x, y) = a_{1,2}(A_1^{2,0} - 2A_1^{1,0} + A_1^{0,0})(x, y) \\ + a_{2,2}(-A_2^{1,0} + A_2^{2,0})(x, y). \end{aligned}$$

In particular, on the triangle λ_3 with vertices $(0, 0)$, $(0, 1)$, and $(1, 0)$, we have, for some constant c,

$$\begin{aligned} g(x, y) &:= f(x, y) - cf_2^0(x, y) \\ &= f(x, y) - a_{1,2}(A_1^{2,0} - 2A_1^{2,0} + A_1^{0,0})(x, y). \end{aligned}$$

Hence, $g \in S_0$ with $\operatorname{supp}(g) \subset \{\tilde{d}_1, ..., \tilde{d}_8\} \subset \{0, 2, 0, 2, 0, 2, 0, 2\}$ such that $\tilde{d}_1 \geqslant 1$. This contradicts the above conclusion, unless $g \equiv 0$, of $f \equiv cf_2^0$.

Next, we will prove that f_3^0 is qms. Let $f \in S_0$ such that

$$\operatorname{supp}(f) \subset \{d_1, ..., d_8\} \subset \{2, 0, 2, 0, 2, 0\ 2, 0\}$$

which is the support of f_3^0. Then from this inclusion property and the property that $d_1 + d_2 \geqslant 2, ..., d_8 + d_1 \geqslant 2$, it is not difficult to verify that we have either $(d_1, d_2) = (2, 0)$ or $(d_1, d_2) = (0, 2)$. If $(d_1, d_2) = (0, 2)$, then by the same argument, we must have $d_8 = 2$, $(d_7, d_6) = (0, 2)$, so that $\operatorname{supp}(f) \subset \{0, 2, 0, 2, 0, 2, 0, 2\} = \operatorname{supp}(f_2^0)$, and f must be a constant multiple of f_2^0. Now suppose that $(d_1, d_2) = (2, 0)$. Then from the linear system (1), we have

$$a_{1,0} = a_{2,0} = 0, \qquad a_{2,-1} = -a_{2,-2}, \qquad a_{3,0} = a_{3,-2},$$
$$a_{3,-1} = -2a_{3,-2},$$

so that for $(x, y) \in \Lambda$, we have

$$\begin{aligned} f(x, y) = a_{2,-2}(-A_2^{-1,1} + A_2^{-2,2})(x, y) \\ + a_{3,-2}(A_3^{0,0} - 2A_3^{-1,1} + A_3^{-2,2})(x, y). \end{aligned}$$

In particular, for (x, y) in the triangle λ_4 with vertices $(-1, 1)$, $(0, 0)$, $(1, 1)$, we have

$$f(x, y) = a_{3,-2}(A_3^{0,0} - 2A_3^{-1,1} + A_3^{-2,2})(x, y)$$

and it follows that

$$g(x, y) := f(x, y) - c_1 f_3^0(x, y)$$

vanishes on this triangle, where c_1 is some constant. Now, by using a similar argument, we have $\operatorname{supp}(g) \subset \{0, 2, 0, 2, 0, 2, 0, 2\}$, so that

$$g = c_2 f_2^0, \qquad \text{or} \qquad f = c_1 f_3^0 + c_2 f_2^0.$$

Hence, if $\operatorname{supp}(f)$ is properly contained in $\operatorname{supp}(f_3^0)$, then $c_1 = 0$ or $f = c_2 f_2^0$, where f_2^0 is ms. This conclusion also implies that f_2^0 is the only ms spline with $\operatorname{supp}(f_2^0) \subseteq \operatorname{supp}(f_3^0)$, so that f_3^0 cannot be written as a linear combination of ms splines in S_0. That is, f_3^0 is qms. We remark, in addition, that if f is qms with $\operatorname{supp} f \subseteq \operatorname{supp}(f_3^0)$, then f is unique in the sense that

$$f = c_1 f_3^0 + c_2 f_2^0, \qquad c_1 \neq 0.$$

From the Bézier represenation of f_2^0 and f_3^0 (cf. Fig. 2), it is clear that f cannot vanish anywhere inside $\operatorname{supp}(f_3^0)$. This shows that $\operatorname{supp}(f)$ must be a convex set.

We now consider the question of uniqueness. Let g be an ls function in S_0 such that $\operatorname{supp}(g)$ is a convex set. Since no vertex of $\operatorname{supp}(g)$ lies in the intersection of only two grid lines, we have

$$\operatorname{supp}(g) = \{d_1, ..., d_8\},$$

where $d_1, ..., d_8$ are nonnegative integers. From the proof of Lemma 4, we have

$$\begin{aligned} &d_1 + d_2 \geqslant 2, ..., d_8 + d_1 \geqslant 2, \\ &d_1 + d_2 + d_3 = d_5 + d_6 + d_7 \\ &d_7 + d_8 + d_1 = d_3 + d_4 + d_5. \end{aligned} \tag{3}$$

Suppose that both

$$d_1 + d_2 + d_3 \geqslant 4$$

and

$$d_7 + d_8 + d_1 \geqslant 4$$

are satisfied. Then it is easy to verify that supp(g) must contain one of the following 8-tuple regions:

$$\{0, 2, 2, 0, 2, 0, 2, 2\}, \qquad \{1, 1, 2, 0, 2, 0, 2, 1\},$$
$$\{2, 0, 2, 0, 2, 0, 2, 0\}, \qquad \{3, 1, 1, 1, 3, 0, 2, 0\}.$$

Since each of these four regions contains the support of a translate of f_3^0, g is not minimal supported. In addition, the same observation implies that if g is qms, supp(g) must be the support of a translate f_3^0, namely, $\{2, 0, 2, 0, 2, 0, 2, 0\}$. The above argument, therefore, implies that

$$g = c_1 f_{3,j}^0 + c_2 f_{2,j}^0, \qquad c_1 \neq 0,$$

for some $j \in Z^2$. This proves the "uniqueness" of f_3^0.

Suppose that at least one of $(d_1 + d_2 + d_3)$ and $(d_7 + d_8 + d_1)$ is smaller than 4. By symmetry of the mesh, we may assume, without loss of generality, that $d_1 + d_2 + d_3 \leqslant 3$. Using (3), we then have

$$1 \leqslant d_2, d_6 \leqslant 3,$$
$$0 \leqslant d_1, d_3, d_5, d_7 \leqslant 1,$$

and consequently,

$$d_4, d_8 \geqslant 1.$$

It follows that supp(g) is a finite union of some supp($f_{1,i}^0$) and supp($f_{2,j}^0$), where $i,j \in Z^2$. Hence, if g is ms, we must have supp(g) = supp($f_{1,i}^0$) or supp($f_{2,j}^0$); the above conclusion implies that g is a constant multiple of $f_{1,i}^0$ or $f_{2,j}^0$. This completes the proof of the proposition.

4. Results on S_r, $r \geqslant 1$

Let M be the ms spline in $S_2^1(\Delta)$ first introduced by Zwart [17], Powell [14], and Powell and Sabin [15] independently, and later studied in detail in [10]. Its support and Bézier representation are given in Fig. 3. We now construct f_i^r, $i = 1, 2, 3$, by convolution, namely,

$$f_i^r = f_i^0 * \underbrace{M * \cdots * M}_{r}.$$

As before, let T_i^r, $i = 1, 2, 3$, be the collection of all constant multiples of translates of f_i^r. It is clear that

$$\text{supp}(f_1^r) = \{r+1, r+1, r+1, r+1, r+1, r+1, r+1, r+1\}$$
$$\text{supp}(f_2^r) = \{r, r+2, r, r+2, r, r+2, r, r+2\}$$

and

$$\text{supp}(f_3^r) = \{r+2, r, r+2, r, r+2, r, r+2, r\}.$$

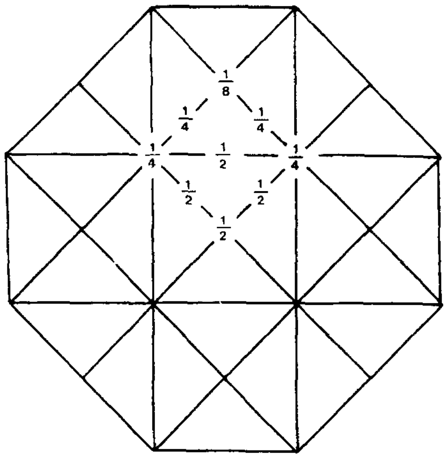


FIGURE 3

We will see that f_1^r and f_2^r are ms functions in S_r and f_3^r is a qms function in S_r. The following lemma is needed.

LEMMA 5. *The supports of the nontrivial ls functions in S_r with smallest 8-tuple supports are*

$$\{r+1, r+1, r+1, r+1, r+1, r+1, r+1, r+1\}$$
$$\{r, r+2, r, r+2, r, r+2, r, r+2\}$$
$$\{r+2, r, r+2, r, r+2, r, r+2, r\}.$$

Proof. Let $\{d_1, ..., d_8\}$ be an 8-tuple support of a nontrivial ls function f in S_r. We first determine the smallest d_1, d_2, d_3, so that by a rotation, the other integers can be determined. The procedure of our proof follows that of Lemma 4 and we use the same notation such as $\hat{\Omega}$, Λ, λ_1, and λ_2. Hence, an expression of $D^r f$ which is similar to that of s in the proof of Lemma 5 is obtained. Here, we recall that $D = D_1^3 D_2 - D_1 D_2^3$. Applying the operator J^r, we have

$$\begin{aligned} f(x, y) &= J^r D^r f(x, y) \\ &= (a_{d_2} A_{1,r}^{d_2,0} + \cdots + a_0 A_{1,r}^{0,0} + a_{-1} A_{1,r}^{-1,1} + \cdots + a_{-d_1} A_{1,r}^{-d_1,d_1} \\ &\quad + b_{d_2} A_{2,r}^{d_2,0} + \cdots + b_0 A_{2,r}^{0,0} + b_{-1} A_{2,r}^{-1,1} + \cdots + b_{-d_1} A_{2,r}^{-d_1,d_1} \\ &\quad + c_{d_2} A_{3,r}^{d_2,0} + \cdots + c_0 A_{3,r}^{0,0} + c_{-1} A_{3,r}^{-1,1} + \cdots + c_{-d_1} A_{3,r}^{-d_1,d})(x, y) \end{aligned}$$

for all $(x, y) \in A$. Since the above expression is obtained by using operator J^r and $A_{i,r} = J^r A_i$, and f vanishes on $\lambda_1 \cup \lambda_2$, it can be seen that the following identities hold,

$$
\begin{aligned}
&a_{d_2} A_{1,r}^{d_2,0} + \cdots + a_0 A_{1,r}^{0,0} = 0\\
&a_0 A_{1,r}^{0,0} + \cdots + a_{-d_1} A_{1,r}^{-d_1,d_1} = 0\\
&b_{d_2} A_{2,r}^{d_2,0} + \cdots + b_0 A_{2,r}^{0,0} = 0\\
&b_0 A_{2,r}^{0,0} + \cdots + b_{-d_1} A_{2,r}^{-d_1,d_1} = 0\\
&c_{d_2} A_{3,r}^{d_2,0} + \cdots + c_0 A_{3,r}^{0,0} = 0\\
&c_0 A_{3,r}^{0,0} + \cdots + c_{-d_1} A_{3,r}^{-d_1,d_1} = 0
\end{aligned}
$$

for all $(x, y) \in \lambda_1 \cup \lambda_2$. Observe, however, that

$$
\begin{aligned}
&A_{1,r}^{i,0}(x, y) = (x-i)^{r+1} y^{3r+3} + o(y^{3r+3}), && (x, y) \in \lambda_1\\
&A_{1,r}^{-i,i}(x, y) = (x+i)^{r-1}(x+y)^{3r+5} + o((x+y)^{3r+5}), && (x, y) \in \lambda_2\\
&A_{2,r}^{i,0}(x, y) = (x-i)^{r} y^{3r+4} + o(y^{3r+4}), && (x, y) \in \lambda_1\\
&A_{2,r}^{-i,i}(x, y) = (x+i)^{r}(x+y)^{3r+4} + o((x+y)^{3r+4}), && (x, y) \in \lambda_2\\
&A_{3,r}^{i,0}(x, y) = (x-i)^{r-1} y^{3r+5} + o(y^{3r+5}), && (x, y) \in \lambda_1
\end{aligned}
$$

and

$$
A_{3,r}^{-i,i}(x, y) = (x+i)^{r+1}(x+y)^{3r+3} + o((x+y)^{3r+3}), \qquad (x, y) \in \lambda_2,
$$

where $o(z^s)$ denotes a term with a factor z^t where the largest t is smaller than s. Hence, the three pairs of identities we obtained above become

$$
\begin{aligned}
&\sum_{i=0}^{d_2} a_i (x-i)^{r+1} = 0\\
&\sum_{i=0}^{d_1} a_{-i} (x+i)^{r-1} = 0\\
&\sum_{i=0}^{d_2} b_i (x-i)^{r} = 0\\
&\sum_{i=0}^{d_1} b_{-i} (x-i)^{r} = 0\\
&\sum_{i=0}^{d_2} c_i (x-i)^{r-1} = 0\\
&\sum_{i=0}^{d_1} c_{-i} (x+i)^{r+1} = 0.
\end{aligned}
$$

It now follows that the smallest pairs (d_1, d_2) that allow nontrivial solutions of $\{a_i, b_i, c_i\}$ in these three pairs of identities must be $(r+1, r+1)$, $(r, r+2)$, and $(r+2, r)$. Similarly, by using the spanning set F^r, the smallest pairs (d_2, d_3) are $(r+1, r+1)$, $(r+2, r)$, and $(r, r+2)$. This completes the proof of the lemma.

We are now ready to establish the main theorem of the paper.

THEOREM 1. *Let $r \geqslant 0$. Then f_1^r and f_2^r are ms functions in S_r and f_3^r is a qms function in S_r. Furthermore, f_1^r, f_2^r, and f_3^r are unique among functions with convex supports in the sense that if f is cms in S_r, then $f \in T_1^r \cup T_2^r$, and if f is cqms in S_r, then $f - g \in T_3^r$ for some $g \in T_2^r$ with* $\operatorname{supp}(g) \subset \operatorname{supp}(f)$. *In addition, f_1^r, f_2^r, and f_3^r together form a (positive) partition of unity; that is, $f_1^r, f_2^r, f_3^r \geqslant 0$ and*

$$\sum_{j \in Z^2} \sum_{i=1}^{3} f_i^r(\cdot - j) \equiv 1. \tag{4}$$

Proof. The proof follows almost exactly as that of Proposition 1. For instance, in proving that f_2^r is ms, the only required change is extending λ_3 to the set $\operatorname{supp}(f_2^r) \cap \{(x, y)\colon 0 \leqslant x + y \leqslant 1\}$, and in proving that f_3^r is qms, the required change is replacing λ_4 by the set $\operatorname{supp}(f_3^r) \cap \{(x, y)\colon 0 \leqslant y \leqslant 1\}$.

To study the question of uniqueness, (3) must be replaced by

$$\begin{aligned} &d_1 + d_2 \geqslant 2r + 2, \ldots, d_8 + d_1 \geqslant 2r + 2 \\ &d_1 + d_2 + d_3 = d_5 + d_6 + d_7 \\ &d_7 + d_8 + d_1 = d_3 + d_4 + d_5. \end{aligned} \tag{5}$$

Then it can be proved that if the inequalities

$$d_1 + d_2 + d_3 \geqslant 3r + 4$$

and

$$d_7 + d_8 + d_1 \geqslant 3r + 4$$

are satisfied, the 8-tuple $\{d_1, \ldots, d_8\}$ which was assumed to be the support of an ls function g in S_r must contain one of the following four 8-tuple regions:

$$\begin{aligned} &\{r, r+2, r+2, r, r+2, r, r+2, r+2\} \\ &\{r+1, r+1, r+2, r, r+2, r, r+2, r+1\} \\ &\{r+2, r, r+2, r, r+2, r, r+2, r\} \\ &\{r+3, r+1, r+1, r+1, r+3, r, r+2, r\}. \end{aligned}$$

Since each of these four regions contains supp(h) for some $h \in T_3^r$, g is not ms. In addition, if g is qms, the support of g must be the support of a translate of f_3^r, namely the third 8-tuple support listed above. This proves "uniqueness" of f_3^r.

If $d_1 + d_2 + d_3 \leqslant 3r+3$ or $d_7 + d_8 + d_1 \leqslant 3r+3$, then (5) implies $d_2, d_4, d_6, d_8 \geqslant r+1$. The same argument as that in the proof of Proposition 1 yields that if g is ms, then $\text{supp}(g) = \text{supp}(f_{1,i}^r)$ or $\text{supp}(f_{2,i}^r)$, where the second subscript i denotes some translate by $i \in Z^2$.

The prove the identity (4), we recall that the special case $r=0$ was already obtained in [7] (cf. [7, Theorem 2]). Since it can be verified that $M^r * 1 \equiv 1$, the proof of the theorem is completed.

Next, we discuss the problem of local basis. Analogous to a result of de Boor and Höllig (cf. [4, Proposition 4.2] and [5, Theorem 2]), we have the following. Since the proof is similar to the one given in [3], we do not include it here.

PROPOSITION 2. *Let G be a closed convex set. Then the collection* $\{f_{1,i}^r, f_{2,i}^r, f_{3,i}^r : i \in Z^2 \cap G\}$ *spans the space*

$$\{f \in S_r : \text{supp}(f) \subset G\}.$$

Using this result, we are now ready to prove the following.

THEOREM 2. $lS_r(\Omega) = S_r(\Omega)$ *if and only if* $r = 0, 1$.

Proof. Suppose that $lS_r(\Omega) = S_r(\Omega)$. Then the above proposition implies that $S_r(\Omega)$ is spanned by

$$\{f \in T_1^r \cup T_2^r \cup T_3^r : \text{supp}(f) \cap \Omega \neq \varnothing\}.$$

Hence, dim $S_r(\Omega)$ does not exceed the dimension of this spanning set, which in turn, does not exceed the upper bound

$$\begin{aligned}(n_1 + 3r + 1)(n_2 + 3r + 1) &+ (n_1 + 3r + 2)(n_2 + 3r + 2) \\ &+ (n_1 + 3r + 3)(n_2 + 3r + 3) = 3n_1 n_2 + (9r + 6)(n_1 + n_2) \\ &+ (3r+1)^2 + (3r+2)^2 + (3r+3)^2.\end{aligned}$$

This yields the inequality

$$3\binom{r+3}{2} - 3 \leqslant 9r + 6$$

or $r^2 - r \leqslant 0$, so that $r = 0, 1$.

Conversely, assume that $r=0$ or 1, so that

$$3(r+2)>(4r+4). \tag{6}$$

Let $P=\operatorname{span}\{p^{l,s}_{1,i,j}, ..., p^{l,s}_{4,i,j}\}$, where $3r+2<s\leqslant l\leqslant 4r+4$, $(i,j)\in Z^2$,

$$p^{l,s}_{1,i,j}(x,y)=(x-i)^{l-s}\,y^s_+$$
$$p^{l,s}_{2,i,j}(x,y)=(x+y-i-j)^{l-s}(y-x-j+i)^s_+$$
$$p^{l,s}_{3,i,j}(x,y)=(y-j)^{l-s}\,x^s_+$$

and

$$p^{l,s}_{4,i,j}(x,y)=(x-y-i-j)^{l-s}(x+y-i+j)^s_+.$$

In addition, let $B=\operatorname{span}\{b(\cdot-(i,j)\mid X^r_k)\}$, where $k=1,2,3$, $(i,j)\in Z^2$, and $b(\cdot\mid X^r_k)$ is the box spline with directions X^r_k, defined as

$$X^r_1=\{\underbrace{e_1,...,e_1}_{r+2},\underbrace{e_1+e_2,...,e_1+e_2}_{r+2},\underbrace{e_2,...,e_2}_{r+2},\underbrace{e_2-e_1,...,e_2-e_1}_{r}\}$$

$$X^r_2=\{\underbrace{e_1,...,e_1}_{r+1},\underbrace{e_1+e_2,...,e_1+e_2}_{r+2},\underbrace{e_2,...,e_2}_{r+2},\underbrace{e_2-e_1,...,e_2-e_1}_{r+1}\}$$

$$X^r_3=\{\underbrace{e_1,...,e_1}_{r},\underbrace{e_1+e_2,...,e_1+e_2}_{r+2},\underbrace{e_2,...,e_2}_{r+2},\underbrace{e_2-e_1,...,e_2-e_1}_{r+2}\}.$$

It is well known that $B=lS_r(\Omega)$ (cf. [3]) and

$$S_r(\Omega)=\pi_{4r+4}+P+B$$

(cf. [1,4, 12]). Hence, it is sufficient to prove that both π_{4r+4} and P are contained in B. It has already been proved by Jia [13] and Bamberger [1] that

$$\max\{m\colon \pi_{m-1}\subset B\}=\min\{3r+6, 4r+5\}.$$

Thus., in view of (6), we have $\pi_{4r+4}\subset B$. To show that $P\subset B$, it is sufficient to prove

$$p^{l,s}_{k,i,j}\in B, \qquad 3r+2<s\leqslant l\leqslant 4r+4, \quad (i,j)\in Z^2, \quad k=1,...,4. \tag{7}$$

We only verify this fact for $k=1$. Set

$$K^l_{1,i,j}=\colon \operatorname{span}\{p^{l,s}_{1,i,j}\colon 3r+2<s\leqslant l\},$$

where $l\leqslant 4r+4$. We first show that

$$K^{4r+4}_{1,0,0}\subset B.$$

Let $C(\cdot \mid X_i^r)$ denote the cone splines (sometimes also called truncated powers) with directions X_i^r and vertex at the origin. Following de Boor and DeVore [2] and de Boor and Höllig [4], we consider the linear map $L_{a,b}$ from span$\{C(\cdot \mid X_i^r)$: $i=1,2,3\}$ into the corresponding univariate spline space $S_{u,r}=S_{u,r}(t_1, ..., t_4)$, where $a, b \in \mathbb{R}^2$, defined by

$$(L_{a,b} f)(t)=f(a+bt), \qquad t \in \mathbb{R}.$$

Of course, the break points $t_1, ..., t_4$ are determined by the intersection of the vector $a+bt$ with the grid lines $e_1, e_1+e_2, e_2, e_2-e_1$. Now, let

$$M_i^r(t)=(L_{a,b} C(\cdot \mid X_i^r))(t).$$

Then M_i^r are univariate B splines with knots indicated below:

$$M_1^r(\cdot)=M_1^r(\cdot \mid \underbrace{t_1, ..., t_1}_{r+2}, \underbrace{t_2, ..., t_2}_{r+2}, \underbrace{t_3, ..., t_3}_{r+2}, \underbrace{t_4, ..., t_4}_{r})$$

$$M_2^r(\cdot)=M_2^r(\cdot \mid \underbrace{t_1, ..., t_1}_{r+1}, \underbrace{t_2, ..., t_2}_{r+2}, \underbrace{t_3, ..., t_3}_{r+2}, \underbrace{t_4, ..., t_4}_{r+1})$$

$$M_3^r(\cdot)=M_3^r(\cdot \mid \underbrace{t_1, ..., t_1}_{r}, \underbrace{t_2, ..., t_2}_{r+2}, \underbrace{t_3, ..., t_3}_{r+2}, \underbrace{t_4, ..., t_4}_{r+2}).$$

The space $S_{u,r}|_{[t_1,t_2]}$ of restrictions of functions $(L_{a,b} f)(t)$, with f in $K_{1,0,0}^{4r+4}$, on the interval $[t_1, t_2]$ has dimension $r+2$. Hence, $S_{u,0}|_{[t_1,t_2]}$ has basis $\{M_1^0, M_2^0\}$ and $S_{u,1}|_{[t_1,t_2]}$ has basis $\{M_1^1, M_2^1, M_3^1\}$. Since both the cone splines $C(\cdot \mid X_i^r)$ and the functions in $K_{1,0,0}^{4r+4}$ are homogeneous functions of degree $4r+4$, we conclude that for each f in $K_{1,0,0}^{4r+4}$, there exist constants α_i^r ($i=0,1$ if $r=0$, and $i=0,1,2$ if $r=1$) such that

$$f(x, y)-\sum_i \alpha_i^r C(x, y \mid X_i^r)=0$$

on the cone $\{(x, y): o \leqslant y \leqslant x\}$, where $r=0, 1$, so that $f-\sum_i \alpha_i^r C(\cdot \mid X_i^r)$ is supported in the cone $\{(x, y): y \geqslant 0$ and $y \geqslant x\}$. Therefore, by using a well known result in [3], we have $K_{1,0,0}^{4r+4} \subset B$. By using the translation invariance property, we also have $K_{1,i,j}^{4r+4} \subset B$, for $(i, j) \in Z^2$. Since all function in $K_{1,0,0}^{4r+3}$ can be written as linear combinations of functions from $K_{1,i,0}^{4r+4}$, where $i \in Z$, the same argument also yields $K_{1,0,0}^{4r+3} \subset B$, and hence, $K_{1,i,j}^{4r+3} \subset B$, and so forth. This verifies (7) for $k=1$. Hence, $P \subset B$.

5. Quasi-interpolation Formulas

We have already seen from Theorem 1 that the ms and qms splines f_1^r, f_2^r, f_3^r form a partition of unity. To improve the approximation order, it is necessary to construct quasi-interpolation formulas. Since the linear algebra involved in obtaining such formulas is very complicated, we only consider the special case $r=1$ in this paper.

Let α_{ij} and β_{ij} be linear functionals on $C(\Omega)$ defined as

$$\alpha_{ij}(g)=\tfrac{5}{3}g(i-\tfrac{1}{2},j-\tfrac{1}{2}) - \tfrac{1}{6}[g(i-1,j-1)+g(i-1,j)+g(i,j-1)+g(i,j)]$$

and

$$\beta_{ij}(g)=\tfrac{5}{3}g(i,j)-\tfrac{1}{6}[g(i-\tfrac{1}{2},j-\tfrac{1}{2})+g(i-\tfrac{1}{2},j+\tfrac{1}{2}) + g(i+\tfrac{1}{2},j-\tfrac{1}{2})+g(i+\tfrac{1}{2},j+\tfrac{1}{2})].$$

We have the quasi-interpolation formula

$$Q(g)=\sum_{i,j}\alpha_{ij}(g)f_1^0(\cdot-(i,j))+\sum_{i,j}\beta_{ij}(g)(f_2^0(\cdot-(i,j))+f_3^0(\cdot-(i,j))).$$

It can be verified that

$$\partial^{m+n}(Q(g)-g)/\partial x^m\,\partial y^n$$

vanishes at all grid points of the partition Δ for all $g\in\pi_3$ and $m+n\leqslant 2$. This yelds the following

PROPOSITION 3. *$Q(g)=g$ for all g in π_3.*

Let $I=[0,1]$ and for each $g\in C(I^2)$, define

$$(Q_n g)(x,y)=(Qg)(nx,ny).$$

By a standard argument in approximation theory, we obtain the following result.

PROPOSITION 4. *If $g\in C(I^2)$, then*

$$\|Q_n g-g\|_{I^2}\leqslant 6\omega\left(g,\frac{1}{n}\right).$$

If, in addition, $g\in C^k(I^2)$, then

$$\|Q_n g-g\|_{I^2}\leqslant 6\frac{4^k}{k!}\left(\max_{r+s\leqslant k}\|g^{(r,s)}\|_{I^2}\right)n^{-k},$$

where $k=1,2,3,4$.

6. Final Remarks

The techniques introduced in this paper can be extended to the study of minimal and quasi-minimal supported splines on any regular mesh in $\mathbb{R}^s$. In a forthcoming paper, we will study recurrence relations, computational schemes, and other important properties of these functions. It should be emphasized that we have only considered ms and qms functions with convex supports. However, we do not anticipate the existence of ms and qms functions whose supports are not convex in the case of regular grid partitions.

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